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A DYNAMIC PROGRAMMING APPROACH TO A COST-EFFECTIVENESS PROBLEM

By William J. Sacco
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FEBRUARY 1965

U. S. ARMY MATERIEL COMMAND
BALLISTIC RESEARCH LABORATORIES
ABERDEEN PROVING GROUND, MARYLAND

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A DYNAMIC PROGRAMMING APPROACH TO A COST-EFFECTIVENESS PROBLEM

William J. Sacco
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Computing Laboratory

RDT & E Project No. 1P014501A14B

A B E R D E E N   P R O V I N G   G R O U N D ,   M A R Y L A N D

B A L L I S T I C   R E S E A R C H   L A B O R A T O R I E S

MEMORANDUM REPORT NO. 1637

WJSacco/PRSchlegel/bj
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A DYNAMIC PROGRAMMING APPROACH TO A COST-EFFECTIVENESS PROBLEM

ABSTRACT

The selection of an optimal weapon system on a cost-effectiveness basis is formulated as a mathematical programming problem. The problem is solved using the functional equation technique of dynamic programming.

The determination of certain "appropriate" values for the arguments of the functional equations is illustrated by a numerical example. The arguments are obtained in a systematic manner by resorting to the construction of auxiliary, bookkeeping tables.

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INTRODUCTION

In evaluating the relative value of a given weapon system with respect to all other weapon systems, frequently a cost-effectiveness basis is established.

A criterion for judging the effectiveness of weapons systems involves a comparison of the results achieved by the respective weapon systems with the costs of the systems.

Cost may be regarded not only as monetary cost but as the cost of the effort required to produce and use the weapon. The evaluation of weapons based on such a comparison of results with efforts may be done in either of two ways: (1) in the first procedure one fixes the cost of the effort, and endeavors to choose a weapon system or a set of weapon systems which yield a maximum of results; (2) in the second procedure, one fixes the results, e.g., an expectation of destroying several targets, and endeavors to choose a weapon system or systems which achieve this result with a minimum cost. Briefly, form (1) gives maximum results for fixed cost, and form (2) provides for determining the minimum cost required to obtain fixed results. The first procedure was employed for the current study.

STATEMENT OF THE PROBLEM

Suppose that we are to select a weapons system composed of n elements $E_1, E_2, \ldots, E_n$. The cost of the system is not to exceed the amount C. This system is to be used to perform an assignment with a result measured by the function $R(E_1, E_2, \ldots, E_n)$. There exist several alternative choices for each element E_i of the system, i.e., E_i belongs to the set $\{E_{i1}, E_{i2}, \ldots E_{ik(i)}\}$. Associated with each E_{ij} is a corresponding cost c_{ij}. The problem then is

$$\max_{(j_1, j_2, \ldots, j_n)} R(E_{1j_1}, E_{2j_2}, \ldots, E_{nj_n})$$

subject to the constraint

$$c_{1j_1} + c_{2j_2} + \ldots + c_{nj_n} \leq C.$$

Fortunately, for the particular problem under consideration the measure R has a separability property, i.e., $R(E_1, E_2, \ldots, E_n) = g_1(E_1) \cdot g_2(E_2) \ldots \cdot g_n(E_n)$.

We are then able to reformulate the problem as a multistage decision problem and apply the functional equation technique of dynamic programming to obtain a feasible computational scheme. The goal is to reduce the n-dimensional problem to a sequence of one-dimensional problems. The ith stage of the problem will result in a determination of an E_i. To attain this simplification we imbed the problem within a family of similar problems, that is, instead of considering a particular total cost of the weapon system, and a fixed number of elements of the system, we consider an entire family of problems where the cost may assume any value less than C and the number of elements may be any natural number less than n. This approach enables one to obtain vital information about the change in optimal policies as the basic parameters C and n vary. Surprisingly it is easier (computationally) to treat the original problem by consideration of the family of problems.

DYNAMIC PROGRAMMING FORMULATION OF THE PROBLEM

To treat this maximization problem by means of functional equation techniques, we introduce the function $f_k(c)$ defined for $0 \le c \le C$ and $k = 1, 2, \ldots, n$ by the relation

$$f_k(c) = \max_{\{j_1, j_2, \ldots, j_k\}} \left[g_1(E_{1j_1}) \cdot g_2(E_{2j_2}) \cdot \ldots \cdot g_k(E_{kj_k}) \right]$$

where the maximum is to be taken over all vectors $(j_1, j_2, \ldots, j_k)$, such that

$$c_{1j_1} + c_{2j_2} + \ldots + c_{kj_k} \le c.$$

Then $f_k(c)$ represents the maximum value of R associated with a weapon system involving k elements with a total cost of not more than c units. The maximization involved in the above equation can be accomplished in k one-dimensional maximization processes by employing Bellman's Principle of Optimality[1]. Then,

$$f_k(c) = \max_{j_k} \left[g_k(E_{kj_k})\, f_{k-1}\,(c - c_{kj_k}) \right]; \; k = 2, \ldots, n.$$

When $k = 1$,

$$f_1(c) = \max_{j_1} g_1(E_{1j_1}),$$

where

$$c_{1j_1} \leq c.$$

A SPECIAL PROBLEM

A weapon system is to be composed of three units, a gun unit, a projectile unit, and a fire control unit. The weapon system is to be assigned the task of destroying a particular target. The measure R is to be identified with the probability, Q, that the system will destroy the target. Therefore we have at our disposal a set of guns $\{G_1, G_2, \ldots, G_\ell\}$, a set of projectiles $\{P_1, P_2, \ldots, P_m\}$, and a set of fire control units $\{F_1, F_2, \ldots, F_n\}$. Let c_{G_i}, c_{P_j}, and c_{F_k} designate the "costs" associated with the gun G_i, the projectile P_j, and the fire control unit F_ℓ, respectively. The problem is to

$$\max_{\{P_i, G_j, F_k\}} Q(P_i, G_j, F_k)$$

subject to the constraint

$$c_{P_i} + c_{G_j} + c_{F_k} \leq c,$$

where

$$Q(P_i, G_j, F_k) = \frac{A_v N}{S^2},$$

$A_v = A_v(P_i)$ is vulnerable area,

$N = N(G_j)$ is the number of rounds fired,

and

$S = S(F_k)$ is the average of the lateral and vertical standard deviations of the miss distance (ft).

Let

$$g_1 = A_v \ ,$$

$$g_2 = N \ ,$$

$$g_3 = \frac{1}{S^2} \ ,$$

and let

$$f_3(c) = \underset{\{P_i, G_j, F_k\}}{\text{Max}} \ g_1(P_i)\ g_2(G_j)\ g_3(F_k)$$

where

$$c_{P_i} + c_{G_j} + c_{F_k} \le c.$$

Using the results of the previous section we find that

$$f_1(c) = \underset{\{P_i\}}{\text{Max}} \ g_1(P_i) \ , \tag{1}$$

$$c_{P_i} \le c \ ,$$

$$f_2(c) = \underset{\{G_j\}}{\text{Max}} \left[g_2(G_j)\ f_1(c - c_{G_j}) \right] \ , \tag{2}$$

$$c_{G_j} \le c \ ,$$

and

$$f_3(c) = \underset{\{F_k\}}{\text{Max}} \left[g_3(F_k)\ f_2(c - c_{F_k}) \right] \tag{3}$$

$$c_{F_k} \le c \ .$$

AN ILLUSTRATIVE EXAMPLE

Problem: Given a total cost constraint, $C = 22$ units, on the entire system and given the tabular inputs

P	$A_v = g_1(P)$	cost, c_P
1	5	1
2	8	2
3	9	4
4	11	7

G	$N = g_2(G)$	cost, c_G
1	4	3
2	6	4
3	9	6
4	12	9
5	17	10

F	$\frac{1}{S^2} = g_3(F)$	cost, c_F
1	4	2
2	9	5
3	16	10
4	25	17

choose a gun, a projectile, and a fire control unit so that the quantity $A_v N/S^2$ is a maximum. At this point the serious reader is encouraged to compute the $f_i(c)$ tables directly from the recurrence relations (1), (2), and (3) using the given input data.

In doing so he will observe that one is frequently plagued by unnecessary computations. That is, he finds that integer unit increases in the value of the argument c do not always produce increases in the payoff function $f_i(c)$, inasmuch as the increase in the value of c is not sufficiently large enough to admit the consideration of an additional entity (another gun or projectile or fire control unit). The method outlined here provides a remedy. As the cost variable is allowed to increase, only those values of c which result in an increase in the payoff function are examined by employing a bookkeeping procedure in the form of auxiliary tables. Using Equation (1), we shall compute and record (in the table below) the values of $f_1(z)$ and $P(z)$ associated with the cost z.

i	$z_i^{(1)}$	$f_1(z_i^{(1)})$	$P(z_i^{(1)})$
1	1	5	1
2	2	8	2
3	4	9	3
4	7	11	4

The next step in the procedure is the computation of $f_2(z)$. Here we encounter the difficulty mentioned above. We shall resort to the construction of several auxiliary, bookkeeping Tables II_1, II_2, ..., II_{12}. Each table will yield at most one row entry of the $f_2(z)$ table.

The method of construction of the bookkeeping tables is now explained. Table II_1 is exhibited.

II_1:

$\bar{c}_G^{(2)}$	$X_G^{(1)}$	$g_2(G)\cdot f_1(X_G^{(1)})$	G
4	$z_1^{(1)} = 1$	20	1
5	$z_1^{(1)} = 1$	30	2
7	$z_1^{(1)} = 1$	45	3
10	$z_1^{(1)} = 1$	60	4
11	$z_1^{(1)} = 1$	85	5

,

where $\bar{c}_G^{(2)} = c_G + X_G^{(1)}$; $G = 1, 2, 3, 4, 5$.

Table II_{i+1} is obtained from Table II_i using the following steps:

1) The G column for Table II_{i+1} is the same as the G column for Table II_i.

2) The entries in the $X_G^{(1)}$ column associated with the minimum values in the $\bar{c}_G^{(2)}$ column of Table II_i are replaced by $z_{i+1}^{(1)}$ from the $f_1(z)$ table. If $z_i^{(1)}$ was the last entry, then $z_{i+1}^{(1)}$ is replaced by a large number. We will use the symbol ∞ to emphasize this fact. Now the entries of the $X_G^{(1)}$ column of Table II_{i+1} are the same as the adjusted entries of the $X_G^{(1)}$ column of Table II_i.

3) The entries in the $\bar{c}_G^{(2)}$ column of II_{i+1} equal $c_G^{(2)} + X_G^{(1)}$, where $G = 1, 2, 3, 4, 5$ and $X_G^{(1)}$ is an entry from the II_{i+1} table.

4) Compute $g_2(G)\cdot f_1(X_G^{(1)})$.

The resulting tables are

$II_2 =$

$\bar{c}_G^{(2)}$	$X_G^{(1)}$	$g_2(G)\cdot f_1(X_G^{(1)})$	G
5	2	32	1
5	1	30	2
7	1	45	3
10	1	60	4
11	1	85	5

$II_3 =$

$\bar{c}_G^{(2)}$	$X_G^{(1)}$	$g_2(G)\cdot f_1(X_G^{(1)})$	G
7	4	36	1
6	2	48	2
7	1	45	3
10	1	60	4
11	1	85	5

$II_4 =$

$\bar{c}_G^{(2)}$	$X_G^{(1)}$	$g_2(G)\cdot f_1(X_G^{(1)})$	G
7	4	36	1
8	4	54	2
7	1	45	3
10	1	60	4
11	1	85	5

$II_5 =$

$\bar{c}_G^{(2)}$	$X_G^{(1)}$	$g_2(G)\cdot f_1(X_G^{(1)})$	G
10	7	44	1
8	4	54	2
8	2	72	3
10	1	60	4
11	1	85	5

$II_6 =$

$\bar{c}_G^{(2)}$	$X_G^{(1)}$	$g_2(G)\cdot f_1(X_G^{(1)})$	G
10	7	44	1
11	7	66	2
10	4	81	3
10	1	60	4
11	1	85	5

$II_7 =$

$\bar{c}_G^{(2)}$	$X_G^{(1)}$	$g_2(G)\cdot f_1(X_G^{(1)})$	G
∞	∞	∞	1
11	7	66	2
13	7	99	3
11	2	96	4
11	1	85	5

$II_8 =$

$\bar{c}_G^{(2)}$	$X_G^{(1)}$	$g_2(G)\cdot f_1(X_G^{(1)})$	G
∞	∞	∞	1
∞	∞	∞	2
13	7	99	3
13	4	108	4
12	2	136	5

$II_9 =$

$\bar{c}_G^{(2)}$	$X_G^{(1)}$	$g_2(G)\cdot f_1(X_G^{(1)})$	G
∞	∞	∞	1
∞	∞	∞	2
13	7	99	3
13	4	108	4
14	4	162	5

$II_{10} =$

$\bar{c}_G^{(2)}$	$X_G^{(1)}$	$g_2(G)\cdot f_1(X_G^{(1)})$	G
∞	∞	∞	1
∞	∞	∞	2
∞	∞	∞	3
16	7	132	4
14	4	153	5

$II_{11} =$

$\bar{c}_G^{(2)}$	$X_G^{(1)}$	$g_2(G)\cdot f_1(X_G^{(1)})$	G
∞	∞	∞	1
∞	∞	∞	2
∞	∞	∞	3
16	7	132	4
17	7	185	5

$\bar{c}_G^{(2)}$	$X_G^{(1)}$	$g_2(G)\cdot f_1(X_G^{(1)})$	G
∞	∞	∞	1
∞	∞	∞	2
∞	∞	∞	3
∞	∞	∞	4
17	7	187	5

$II_{12} =$ (table above)

We will now obtain the entries for the $f_2(z)$ table from the auxiliary tables in the following manner:

For each table II_i let

$$z_i^{(2)} = \min_G \left\{\bar{c}_G^{(2)}\right\}. \quad \text{Then}$$

$$f_2(z_1^{(2)}) = \max_G \left\{g_2(G)\cdot f_1(X_G^{(1)}) \ / \ \bar{c}_G^{(2)} = z_1^{(2)}\right\};$$

$$f_2(z_i^{(2)}) = \max \left\{f_2(z_{i-1}^{(2)}), \max_G \left\{g_2(G)\cdot f_1(X_G^{(1)}) \ / \ \bar{c}_G^{(2)} = z_i^{(2)}\right\}\right\},$$

for $i > 1$.

From our example we obtain $z_1^{(2)} = \min \{4, 5, 7, 10, 11\} = 4$, and $f_2(4) = \max \{g_2(1)\cdot f_1(1)\} = \max \{20\} = 20$. From table II_2 $z_2^{(2)} = \min \{5, 5, 7, 10, 11\} = 5$, and $f_2(5) = \max \{f_2(4), \max \{g_2(1)\cdot f_1(1), g_2(2)\cdot f_1(1)\}\}$

$$= \max \{20, \max \{32, 30\}\} = 32.$$

If we continue in this manner, we obtain the $f_2(z)$ table:

i	$z_i^{(2)}$	$f_2(z_i^{(2)})$	$G(z_i^{(2)})$	$X_G^{(1)}(z_i^{(2)})$
1	4	20	1	1
2	5	32	1	2
3	6	48	2	2
4	8	72	3	2
5	10	81	3	4
6	11	96	4	2
7	12	136	5	2
8	14	153	5	4
9	17	187	5	7

The entries in the $G(z_i^{(2)})$ and $X_G^{(1)}(z_i^{(2)})$ columns are the values of G and $X_G^{(1)}$ respectively, which yield the value of $f_2(z_i^{(2)})$. If $f_2(z_{i+1}^{(2)}) = f_2(z_i^{(2)})$, then $z_{i+1}^{(2)}$ and the values associated with $z_{i+1}^{(2)}$ are not recorded in the table.

We proceed in a similar manner to the construction of the $f_3(z)$ table. The auxiliary tables are

III_1 =

$\bar{c}_F^{(3)}$	$X_F^{(2)}$	$G_3(F)\cdot f_2(X_F^{(2)})$	F
6	4	80	1
9	4	180	2
14	4	320	3
21	4	500	4

, III_2 =

$\bar{c}_F^{(3)}$	$X_F^{(2)}$	$G_3(F)\cdot f_2(X_F^{(2)})$	F
7	5	128	1
9	4	180	2
14	4	320	3
21	4	500	4

,

III_3 =

$\bar{c}_F^{(3)}$	$X_F^{(2)}$	$G_3(F)\cdot f_2(X_F^{(2)})$	F
8	6	192	1
9	4	180	2
14	4	320	3
21	4	500	4

, III_4 =

$\bar{c}_F^{(3)}$	$X_F^{(2)}$	$G_3(F)\cdot f_2(X_F^{(2)})$	F
10	8	288	1
9	4	180	2
14	4	320	3
21	4	500	4

,

$$
III_5 = \begin{array}{c|c|c|c}
\bar{c}_F^{(3)} & X_F^{(2)} & G_3(F)\cdot f_2(X_F^{(2)}) & F \\
\hline
10 & 8 & 288 & 1 \\
10 & 5 & 288 & 2 \\
14 & 4 & 320 & 3 \\
21 & 4 & 500 & 4
\end{array}
, \quad
III_6 = \begin{array}{c|c|c|c}
\bar{c}_F^{(3)} & X_F^{(2)} & G_3(F)\cdot f_2(X_F^{(2)}) & F \\
\hline
12 & 10 & 324 & 1 \\
11 & 6 & 432 & 2 \\
14 & 4 & 320 & 3 \\
21 & 4 & 500 & 4
\end{array}
,
$$

$$
III_7 = \begin{array}{c|c|c|c}
\bar{c}_F^{(3)} & X_F^{(2)} & G_3(F)\cdot f_2(X_F^{(2)}) & F \\
\hline
12 & 10 & 324 & 1 \\
13 & 8 & 648 & 2 \\
14 & 4 & 320 & 3 \\
21 & 4 & 500 & 4
\end{array}
, \quad
III_8 = \begin{array}{c|c|c|c}
\bar{c}_F^{(3)} & X_F^{(2)} & G_3(F)\cdot f_2(X_F^{(2)}) & F \\
\hline
13 & 11 & 384 & 1 \\
13 & 8 & 648 & 2 \\
14 & 4 & 320 & 3 \\
21 & 4 & 500 & 4
\end{array}
,
$$

$$
III_9 = \begin{array}{c|c|c|c}
\bar{c}_F^{(3)} & X_F^{(2)} & G_3(F)\cdot f_2(X_F^{(2)}) & F \\
\hline
14 & 12 & 544 & 1 \\
15 & 10 & 729 & 2 \\
14 & 4 & 320 & 3 \\
21 & 4 & 500 & 4
\end{array}
, \quad
III_{10} = \begin{array}{c|c|c|c}
\bar{c}_F^{(3)} & X_F^{(2)} & G_3(F)\cdot f_2(X_F^{(2)}) & F \\
\hline
16 & 14 & 648 & 1 \\
15 & 10 & 729 & 2 \\
15 & 5 & 512 & 3 \\
21 & 4 & 500 & 4
\end{array}
,
$$

$$
III_{11} = \begin{array}{c|c|c|c}
\bar{c}_F^{(3)} & X_F^{(2)} & G_3(F)\cdot f_2(X_F^{(2)}) & F \\
\hline
16 & 14 & 648 & 1 \\
16 & 11 & 864 & 2 \\
16 & 6 & 768 & 3 \\
21 & 4 & 500 & 4
\end{array}
, \quad
III_{12} = \begin{array}{c|c|c|c}
\bar{c}_F^{(3)} & X_F^{(2)} & G_3(F)\cdot f_2(X_F^{(2)}) & F \\
\hline
19 & 17 & 748 & 1 \\
17 & 12 & 1224 & 2 \\
18 & 8 & 1152 & 3 \\
21 & 4 & 500 & 4
\end{array}
,
$$

$$
III_{13} = \begin{array}{c|c|c|c}
\bar{c}_F^{(3)} & X_F^{(2)} & G_3(F)\cdot f_2(X_F^{(2)}) & F \\
\hline
19 & 17 & 748 & 1 \\
19 & 14 & 1377 & 2 \\
18 & 8 & 1152 & 3 \\
21 & 4 & 500 & 4
\end{array}
, \quad
III_{14} = \begin{array}{c|c|c|c}
\bar{c}_F^{(3)} & X_F^{(2)} & G_3(F)\cdot f_2(X_F^{(2)}) & F \\
\hline
19 & 17 & 748 & 1 \\
19 & 14 & 1377 & 2 \\
20 & 10 & 1296 & 3 \\
21 & 4 & 500 & 4
\end{array}
,
$$

III_{15} =

$\bar{c}_F^{(3)}$	$x_F^{(2)}$	$G_3(F)\cdot f_2(x_F^{(2)})$	F
∞	∞	∞	1
22	17	1683	2
20	10	1296	3
21	4	500	4

, III_{16} =

$\bar{c}_F^{(3)}$	$x_F^{(2)}$	$G_3(F)\cdot f_2(x_F^{(2)})$	F
∞	∞	∞	1
22	17	1683	2
21	11	1536	3
21	4	500	4

,

III_{17} =

$\bar{c}_F^{(3)}$	$x_F^{(2)}$	$G_3(F)\cdot f_2(x_F^{(2)})$	F
∞	∞	∞	1
22	17	1683	2
22	12	2176	3
22	5	800	4

.

The $f_3(z)$ table is

i	$z_i^{(3)}$	$f_3(z_i^{(3)})$	$F(z_i^{(3)})$	$x_F^{(2)}(z_i^{(3)})$
1	6	80	1	4
2	7	128	1	5
3	8	192	1	6
4	10	288	1 2	8 5
5	11	432	2	6
6	13	648	2	8
7	15	729	2	10
8	16	864	2	11
9	17	1224	2	12
10	19	1377	2	14
11	21	1536	3	11
12	22	2176	3	12

.

Now, the $f_i(z)$ tables, $i = 1, 2, 3$, will yield the desired results, that is, the value of the maximum pay-off, subject to the constraint on the total cost, and the policy used to obtain this maximum.

The entries in the tables are interpreted in the following manner:

In the $f_3(z)$ table the column labelled $f_3(\ )$ contains the values of the maximum pay-off, subject to the total cost equaling the entries in $z_i^{(3)}$. In the column labelled $F(\ )$ we obtain the policy F, which yields this maximum; and the value in the column labelled $X_F^{(2)}(z_i^{(3)})$ associated with $z_i^{(3)}$ is the remaining amount to be used for determining the policy G (which will be obtained from the $f_2(z)$ table). From the $f_2(z)$ table, select the value in the $z_i^{(2)}$ column which is equal to the previously determined entry from $X_F^{(2)}(z_i^{(3)})$. The values from $G(z_i^{(2)})$ and from $X_G^{(1)}(z_i^{(2)})$ associated with this entry from $z_i^{(2)}$ are the policy G and value used to determine the policy P from $f_1(z)$ table, respectively.

In the example the allowable cost is 22. Hence, from the $f_3(z)$ table we observe that the maximum pay-off is 2176, that F = 3, and that the amount remaining to be allocated is 12 units. Going to the $f_2(z)$ table, we determine the policy G = 5 with a remaining cost of 2, which yields the policy P = 2 from the $f_1(z)$ table.

Summarizing, we have obtained the optimum policy P = 2, G = 5 and F = 3, which yields

$$A_V N. \frac{1}{S^2} = 8 \cdot 17 \cdot 16 = 2176$$

with a cost of

$$C_P + C_G + C_F = 2 + 10 + 10 = 22.$$

In the general problem the question arises: How many auxiliary tables does one construct at each stage? We can compute an upper bound on the number of tables required. Define

$$k_i = \min_j \{c_{ij}\}, \ i = 1, \ldots, n ;$$

$$K_i = \max_j \{c_{ij}\}, \ i = 1, \ldots, n ;$$

$$M_i = \min \left\{ \sum_{j=1}^{i} K_j, \ C - \sum_{j=i+1}^{n} k_j \right\}, \ i = 1, \ldots, n.$$

The last argument, $z_j^{(i)}$, in the $f_i(z_j^{(i)})$ table is bounded above by M_i, that is, in the ith stage when all the entries in the $\bar{c}_{E_i}^{(i)}$ column exceed M_i, the construction of the auxiliary tables can be halted.

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